Study of Predator-Prey Model with Weak Allee Effect and Pesticide Usage

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1 Introduction

There are several types of predator-prey interactions and some researchers have conducted extensive research on the dynamics of interacting predator-prey models to understand the long-term behavior of species. The population of predator and prey vary with the time. Considering natural prey deaths, prey deaths due to predators, predator survival dependence solely on prey with a natural death rate proportionate to predator population, the mathematical model will look like,

$$\frac{dX}{dt} = aX - C_1XY$$

$$\frac{dY}{dt} = C_2XY - bY$$

where C_1, C_2 are interaction parameters.

This a classic predator-prey model, which was independently proposed by Lotka in the United States in 1925 and Volterra in Italy in 1926. In population dynamics, when the population density is very low, there is a positive correlation between the population unit growth rate and the population density. This phenomenon can be called the Allee effect. A region is said to have a strong Allee effect when population shrinks to lower densities and weak Allee effect when the proliferation rate is increasing and positive due to a limited amount of resources. In this paper we have introduced a predator-prey model with weak alley effect and the impact of usage of a pesticide named Carbofuran, on predator-prey system.

In the above model, we made a assumption that prey grows exponentially even in absence of predator, which is unrealistic. Hence, we add a density-dependent growth with carrying capacity of K. Now with Alley effect and impact of pesticide in conjunction with density-dependent growth our model becomes,

$$\frac{dX}{dt} = aX\left(\frac{X}{A} - 1\right)\left(1 - \frac{X}{K}\right) - P_1X - C_1YX$$

$$\frac{dY}{dt} = C_2XY - P_2X - bY$$

where P_1, P_2 are per-capita death rates due to usage of Carbofuran, the pesticide, and A is the Allee effect constant. When the density is below the critical threshold, the population affected by the strong Allee effect will have a negative average growth rate. Under deterministic dynamics, we find that populations that do not exceed this threshold will be extinct. Most research only considers the strong Allee effect, but in the work of Allee it is clear that the Allee effect also has a weak Allee effect[3, 4, 5]. This research is mainly about weak Alley effect in combination with usage of pesticide. The equations are presented below.

2 Modeling

2.1 Logistic growth predator-prey model

1) Equations: Let us consider prey population with density-dependent growth with carrying capacity of K. The system becomes,

$$\frac{dX}{dt} = aX\left(1 - \frac{X}{K}\right) - C_1YX \tag{1}$$

$$\frac{dY}{dt} = C_2XY - bY \tag{2}$$

2) Nullclines: We now find the nullclines at,

$$\frac{dX}{dt} = \frac{dY}{dt} = 0$$

For $X' = 0$,
1) $X = 0$
2) $a = \frac{aX}{K} + C_1Y$

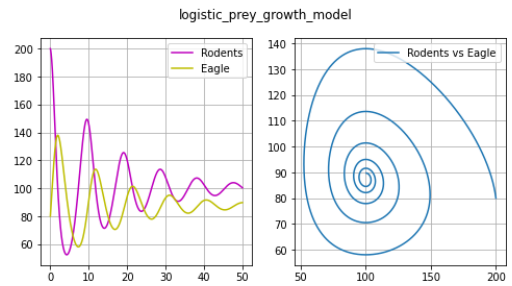
For $Y' = 0$
1)$Y = 0$ or
2)$X = \frac{b}{C_2}$

3) Equilibrium points: We now find equilibrium points for the above 4 nullclines. The equilibrium solutions are,

$$(0, 0)$$

$$(K, 0)$$

$$\left(\frac{b}{C_2}, \frac{a(KC_2 - b)}{KC_1C_2}\right)$$



From the above figure, we can conclude that amplitude of the oscillations decreases with time for both populations and as the time increases, the figure suggests that each population settles to a fixed population density.

2.2 Logistic growth predator-prey with pesticide model

Now due to rodenticide used to either protect prey from Predator or kill prey itself, we observe P_1 and P_2 per capita deaths in both prey and predator. The model then becomes,

1) Equations:

$$\frac{dX}{dt} = aX\left(1 - \frac{X}{K}\right) - C_1YX - P_1X \tag{3}$$

$$\frac{dY}{dt} = C_2XY - (b + P_2)Y \tag{4}$$

2) Nullclines: We get nullclines by equating equations (3) and (4) to 0.

For $X' = 0$,

1)$X = 0$
2)$\frac{aX}{K} + C_1Y = a - P_1$

For $Y' = 0$,

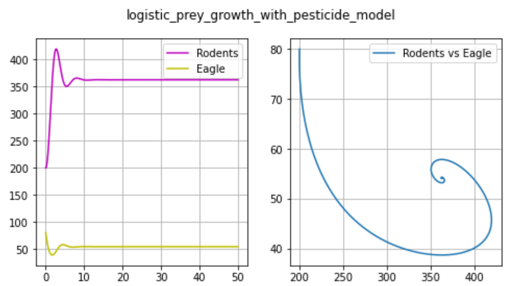
1)$Y = 0$
2)$X = \frac{b+P_2}{C_2}$

3) Equilibrium points: The equilibrium points will be,

$$(0, 0)$$

$$\left(K\left(1 - \frac{P_1}{a}\right), 0\right)$$

$$\left(\frac{b + P_2}{C_2}, \frac{a}{C_1}\left(1 - \frac{b + P_2}{KC_2}\right) - \frac{P_1}{C_1}\right)$$



From the above figure, we can conclude that as the populations oscillates around the equilibrium populations and then settles down to these values. Here particularly, we also see that the equilibrium population of Rodents is more than initial population and that of Eagles decreases.

3 Allee effect

In the density-dependent model, we can see that population can never exceed the carrying capacity (K) that the environment can hold. Even though the population density is very low, we see that there is a exponential growth. This is the flaw of logistic equation. Warder Clyde Allee showed that a population can do much worse when it is very sparse. The reasons include not being able to find suitable mate, needed more help to find food. Due to his work, a population that has a declining per-capita growth rate is set to show an Allee effect. The mathematical representation of this model is,

$$\frac{dX}{dt} = aX\left(\frac{X}{A} - 1\right)\left(1 - \frac{X}{K}\right) \tag{5}$$

where A is the Allee threshold.

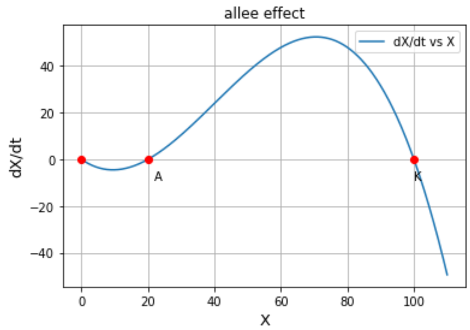
To understand the behavior of this model we find the equilibrium points of equation (5). The equilibrium points are,

$$X = 0$$

$$X = K$$

$$X = A$$

So when population is below A, the rate of change goes negative and it approaches 0. Hence, extinction of a species is possible. Whereas, if population is above A, the rate if change is positive and approaches to carrying capacity K and if its above K, then rate of change is negative and approaches carrying capacity K.



3.1 Logistic growth predator-prey with Weak Allee effect model

Many eagles were found dead in the fields of Maryland, Montana. According to the authorities farmers are using pesticides/rodenticides (like Carbofuran) to kill the animals that will come to eat plants[2], crops and live stock. This in turn kills eagles. In this paper we are introducing a weak Allee effect on rodents. A population of rodents exhibiting a weak Allee effect will possess a reduced per-capita growth rate lower population density. However, even at this low population size, the population of rodents will always exhibit a positive per-capita growth rate.

1) Equations of our model with weak Allee effect:

$$\frac{dX}{dt} = aX\left(1 - \frac{X}{K}\right)\left(\frac{X}{X+A}\right) - C_1YX \tag{6}$$

$$\frac{dY}{dt} = C_2XY - bY \tag{7}$$

where X is population of Rodents and Y is the population of Eagles.

2) Nullclines: (Assuming X, $Y \geq 0$) We get nullclines by equating equations (6) and (7) to 0.

For $X' = 0$,

1) $X = 0$

2) $aX^2 + (C_1KY - aK)X + C_1KAY = 0$

For $Y' = 0$

1) $Y = 0$
2) $X = \frac{b}{C_2}$

3) The equilibrium points will be,

$$(0, 0)$$
$$(K, 0)$$
$$\left(\frac{b}{C_2}, \frac{ab(KC_2 - b)}{C_1 C_2 K(b + AC_2)}\right)$$

Theorem:

1. Trivial equilibrium point E_1 is always a saddle node point.
2. E_2 is stable for $C_2 < \frac{b}{K}$ and is a saddle point otherwise.
3. Coexistence equilibrium E_3 is locally asymptotically stable for $A < \frac{b^2}{C_2(C_2K-2b)}$ and is unstable node otherwise.

The Jacobian Matrix is,

$$J = \begin{pmatrix} -C_1Y + \frac{Xa\left(1-\frac{X}{K}\right)}{A+X} + X\left(-\frac{Xa\left(1-\frac{X}{K}\right)}{(A+X)^2} + \frac{a\left(1-\frac{X}{K}\right)}{A+X} - \frac{Xa}{K(A+X)}\right) & -C_1X \\ C_2Y & C_2X - b \end{pmatrix}$$

Now, let's calculate jacobian for each equilibrium point.

1) Jacobian at equilibrium point E_1 is given as,

$$\begin{pmatrix} 0 & 0 \\ 0 & -b \end{pmatrix}$$

Hence E_1 will be a semi-stable or saddle point.

2) Jacobian at equilibrium point E_2 is given as,

$$\begin{pmatrix} -\frac{Ka}{A+K} & -C_1K \\ 0 & C_2K - b \end{pmatrix}$$

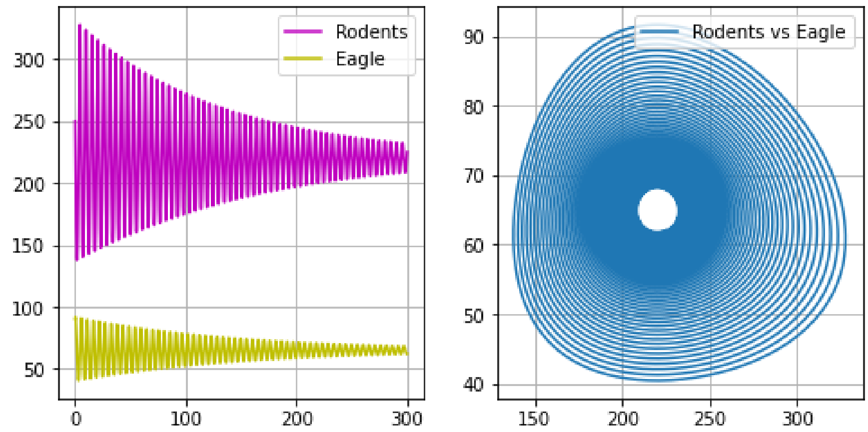
$\frac{-Ka}{A+K}$ is negative, hence E_2 is stable when $C_2K - b < 0$ and is semi-stable or stable when $C_2K - b > 0$.

3) Jacobian at equilibrium point E_3 is given as,

$$\begin{pmatrix} \frac{ab\left(1-\frac{b}{C_2K}\right)}{C_2\left(A+\frac{b}{C_2}\right)} + \frac{b\left(\frac{a\left(1-\frac{b}{C_2K}\right)}{A+\frac{b}{C_2}} - \frac{ab\left(1-\frac{b}{C_2K}\right)}{C_2\left(A+\frac{b}{C_2}\right)^2} - \frac{ab}{C_2K\left(A+\frac{b}{C_2}\right)}\right)}{C_2} - \frac{ab(C_2K-b)}{C_2K(AC_2+b)} & -\frac{C_1b}{C_2} \\ \frac{ab(C_2K-b)}{C_1K(AC_2+b)} & 0 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{-2ax^{*3}+aKx^{*2}-3aAx*2+2aAKx^*}{K(x^*+A)^2} - C_1y^* & \frac{-bC_1}{C_2} \\ C_2y^* & 0 \end{pmatrix} \text{ where } x^* = \frac{b}{C_2} \text{ and } y^* = \frac{ab(KC_2-b)}{C_1C_2K(b+AC_2)}$$

Here trace is $\frac{-ax^*(x^{*2}+2Ax^*-AK)}{K(x^*+A)^2}$ and determinant is $-bC_1y^*$.
Now if Trace < 0 and $A < \frac{x^{*2}}{(K-2x^*)}$, then the positive equilibrium is locally asymptotically stable.
Now if Trace > 0 and $A > \frac{x^{*2}}{(K-2x^*)}$, then the positive equilibrium is unstable.



From the above figure, we can conclude that both Rodents and Eagles are approaching towards equilibrium and due to Allee effect the equilibrium population for both Rodents and Eagles is less than the initial population.

3.2 Predator-Prey model with Weak Allee effect and pesticide

Let us see how our above model changes when both rodents and eagles are infected due to introduction of rodenticide. As mentioned previously, let P_1 and P_2 be per capita deaths in both eagles and rodents because of usage of rodenticide. The model then becomes,

1) Equations of our model with weak Allee effect and rodenticide:

$$\frac{dX}{dt} = aX\left(1 - \frac{X}{K}\right)\left(\frac{X}{X+A}\right) - C_1YX - P_1X \tag{8}$$

$$\frac{dY}{dt} = C_2XY - (b+P_2)Y \tag{9}$$

2) Nullclines: We get nullclines by equating equations (8) and (9) to 0.

For $X' = 0$,
1)$X = 0$
2)$\frac{aX}{X+A} - \frac{aX^2}{K(X+A)} - C_1Y - P_1 = 0$

For $Y' = 0$
1)$Y = 0$
2)$X = \frac{b+P_2}{C_2}$

3) The equilibrium points will be,

$(0,\ 0),\ (\alpha,\ 0),\ (\beta,\ 0),\ (X^*,\ Y^*)$

where,
$\alpha = \frac{-P_1+a-\sqrt{-4AP_1a+P_1^2-2P_1a+a^2}}{2a}$, $\beta = \frac{-P_1+a+\sqrt{-4AP_1a+P_1^2-2P_1a+a^2}}{2a}$ and

$X^* = \frac{P_2+b}{C_2}$, $Y^* = \frac{(-AP_1-P_1X^*-X^{*2}a+X^*a)}{C_1(A+X^*)}$

The Jacobian Matrix is,

$$J = \begin{pmatrix} -C_1Y - P_1 + \frac{Xa\left(1-\frac{X}{K}\right)}{A+X} + X\left(-\frac{Xa\left(1-\frac{X}{K}\right)}{(A+X)^2} + \frac{a\left(1-\frac{X}{K}\right)}{A+X} - \frac{Xa}{K(A+X)}\right) & -C_1X \\ C_2Y & C_2X - P_2 - b \end{pmatrix}$$

1) Jacobian matrix at equilibrium point E_1 is given by,

$$\begin{pmatrix} -P_1 & 0 \\ 0 & -P_2 - b \end{pmatrix}$$

2) Jacobian matrix at equilibrium point E_2 is given by,

$$\begin{pmatrix} \frac{-KP_1(A+\alpha)^2+a\alpha(A+\alpha)(K-\alpha)-a\alpha(\alpha(K-\alpha)+(A+\alpha)(-K+2\alpha))}{K(A+\alpha)^2} & -C_1\alpha \\ 0 & C_2\alpha - P_2 - b \end{pmatrix}$$

3) Jacobian matrix at equilibrium point E_3 is given by,

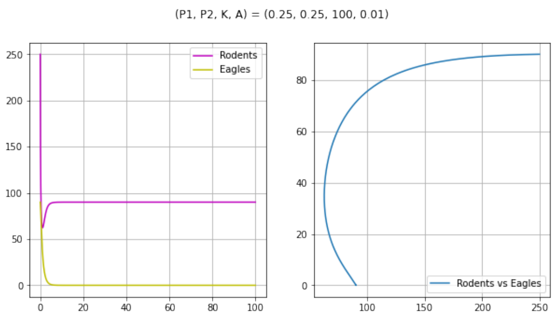
$$\begin{pmatrix} \frac{-KP_1(A+\beta)^2+a\beta(A+\beta)(K-\beta)-a\beta(\beta(K-\beta)+(A+\beta)(-K+2\beta))}{K(A+\beta)^2} & -C_1\beta \\ 0 & C_2\beta - P_2 - b \end{pmatrix}$$

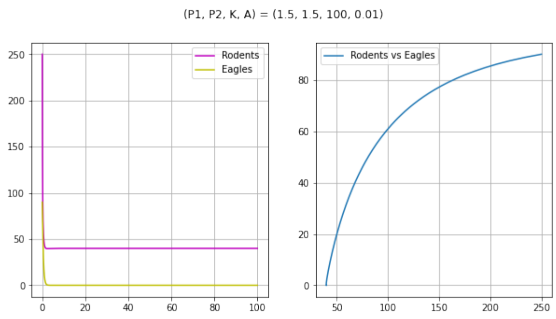
4) Jacobian matrix at equilibrium point E_4 is given by,

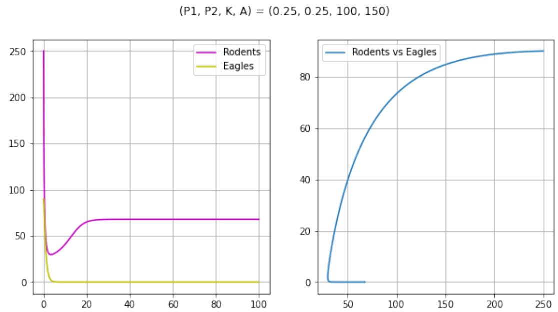
$$\begin{pmatrix} \frac{-K(A+X^*)^2(C_1Y^*+P_1)+X^*a(A+X^*)(K-X^*)-X^*a(X^*(K-X^*)+(A+X^*)(-K+2X^*))}{K(A+X^*)^2} & -C_1X^* \\ C_2Y^* & C_2X^* - P_2 - b \end{pmatrix}$$

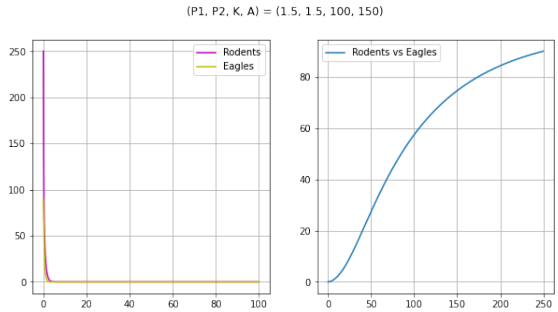
4 Numerical Simulation

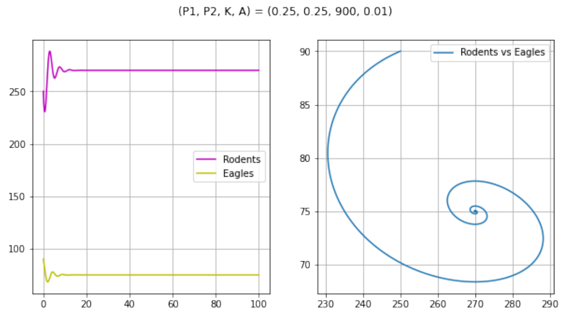
In this section, we present some numerical simulation to illustrate our theoretical analysis.

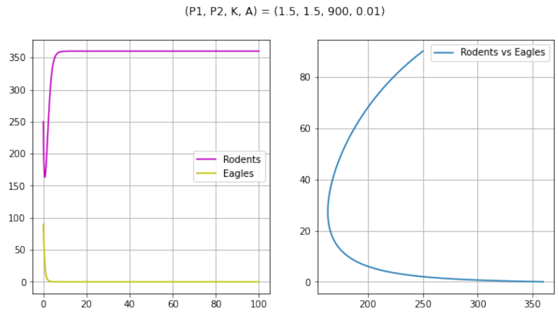


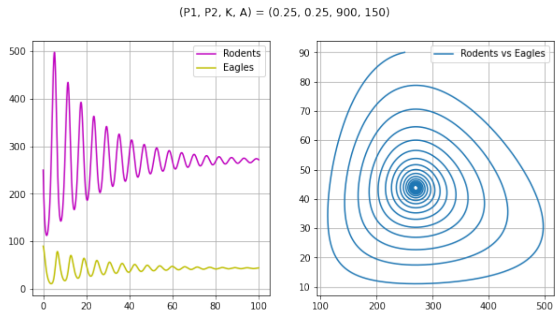


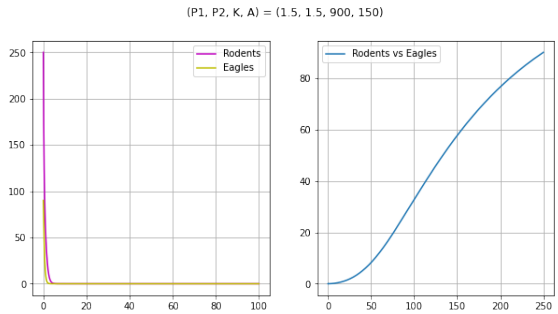












5 Conclusion

The significance of Allee constant A can be clearly seen from the above graphs. Species exhibiting weak Allee effect, will eventually lead to perishing or a significant decrease in equilibrium population for Predators. This is clearly in accordance with what happened with Bald Eagles vs Rodents upon usage of Carbofuran in Washington[1].

From the above research, we can conclude the following points,

1. When the pesticide is added in low amounts,
   - low carrying capacity (K) and any Allee constant (A) $\implies$ predator extinction and prey decrease (but not extinct)
   - high carrying capacity (K) and low Allee constant (A) $\implies$ predator decrease and prey increase and equilibrium obtained relatively faster
   - high carrying capacity (K) and high Allee constant (A) $\implies$ predator decrease and prey increase with coesistence for a longer duration and equilibrium obtained relatively slower

2. When the pesticide is added in high amounts,
   - low carrying capacity (K) and low Allee constant (A) $\implies$ predator extinction and prey decrease (but not extinct)
   - any carrying capacity (K) and high Allee constant (A) $\implies$ both predator and prey extinction
   - high carrying capacity (K) and low Allee constant (A) $\implies$ both predator extinction and prey increase

6 Future scope

We would like to extend this research further by estimating the best values for the parameters involved and comment on the results for the model above using a cost function with both least-squares and cubic-spline techniques and compare their results.

References

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